MATHEMATICS

Paper I : Real Analysis

Note : Attempt **all** questions from **Section-A** (Objective type questions) and **Section-B** (Short answer type questions) and **three** questions from **Section-C** (Long/Essay type questions).

Section—A $1 \times 10 = 10/0{\cdot}5 \times 10 = 5$

1. Give an example of the set which is bounded below but not bounded above.

2. The range of the sequence $< (-1)^n >$ is.........

3. The value of $\underset{x \to 0}{\text{Lt}} (1 + x)^{1/x}$ is :

(a) 0 (b) 1 (c) e (d) $1/e$.

4. Let $f(x) = \frac{1}{x} \sin x$ when $x \neq 0$, then the value of $f(0)$, so that $f(x)$ becomes continuous at $x = 0$ is :

(a) 0 (b) 1 (c) e (d) 2.

5. The maximum product of three positive integers whose sum is 30, will be :

(a) 300 (b) 1000 (c) 3000 (d) 0.

6. A constant function :

(a) R-Integrable (b) not R-Integrable

(c) Sometimes R-Integrable (d) None of these.

7. Integral $\int_1^\infty \frac{dx}{x^n}$ is convergent if :

(a) $\forall n$ (b) $n > 1$ (c) $n < 1$ (d) Never.

8. Define diameter of a set in Metric Space.

9. State Taylor's Theorem for two variables.

10. Define Complete Metric Space.

Section—B $5 \times 2 = 10/5 \times 1 = 5$

1. Show that the sequence $< \sqrt{n^2 + 1} - n >$ is convergent.

2. Show that the sequence $< f_n >$ where $f_n(x) = nx(1 - x)^n$ does not converges uniformly on interval $[0, 1]$.

3. Prove that the function $f(x) = \sin x$ is uniform continuous on $[0, \infty[$.

4. Let $f : \mathrm{R}^2 \to \mathrm{R}$, defined by $f(x) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } x, y \neq 0 \\ 0 & \text{if } x, y = 0 \end{cases}$ then show that $\underset{(x, y) \to (0, 0)}{\text{Lt}} f(x, y)$ does not exist.

5. Expand $f(x, y) = x^2 + xy + y^2$ in terms of $(x - 2)$ and $(y - 3)$ upto IInd order term. **Or**

Find the maximum value of the function $x^p y^q z^r$ subject to condition $ax + by + cz = p + q + r$.

6. Prove that : $\underset{n \to \infty}{\text{Lt}} \left(\frac{n^n}{\underline{|n}} \right)^{1/n} = e$.

7. Define Metric Space. **Or**

Show that in metric space every open sphere in an open set.

Section—C $3 \times 10 = 30/3 \times 5 = 15$

1. If (X, d) be a metric space and 'm' be a the real number, then show that $\rho(x, y) = \frac{md(x, y)}{1 + d(x, y)}$ is a metric on X and diam $(X) \leq m$.

2. Prove that : $\int_0^{\pi/2} \log\left(\frac{a + b\sin\theta}{a - b\sin\theta}\right)\frac{d\theta}{\sin\theta}, a > b = \pi\sin^{-1}\frac{b}{a}$

3. Show that the integral : $\int_0^{\pi/2} \log \sin x dx$ converges. **Or**

If $f(x)$ be defined on $[0, 2]$ as :

$$f(x) = x + x^2 \text{ when } x \text{ is rational}$$
$$= x^2 + x^3 \text{ when } x \text{ is irrational.}$$

then $f(x)$ is not R-integrable on $[0, 2]$.

4. Discuss the continuity of the function at $(0, 0)$:

$$f(x, y) = \frac{xy^3}{x^2 + y^6} \text{ if } x, y \neq 0$$
$$= 0 \quad \text{if } x, y = 0$$

5. Prove that : $\underset{x \to 1}{\text{Lt}} \left[\sum_{n=1}^{\infty} \frac{nx}{n^3 + x^2}\right] = \sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$ **Or**

Show that $f(x) = [x]$, nearest integer $\leq x$ is Reimann integrable on $[0, 5]$ and $\int_0^5 [x]\, dx = 10$.